Betz Limit Derivation
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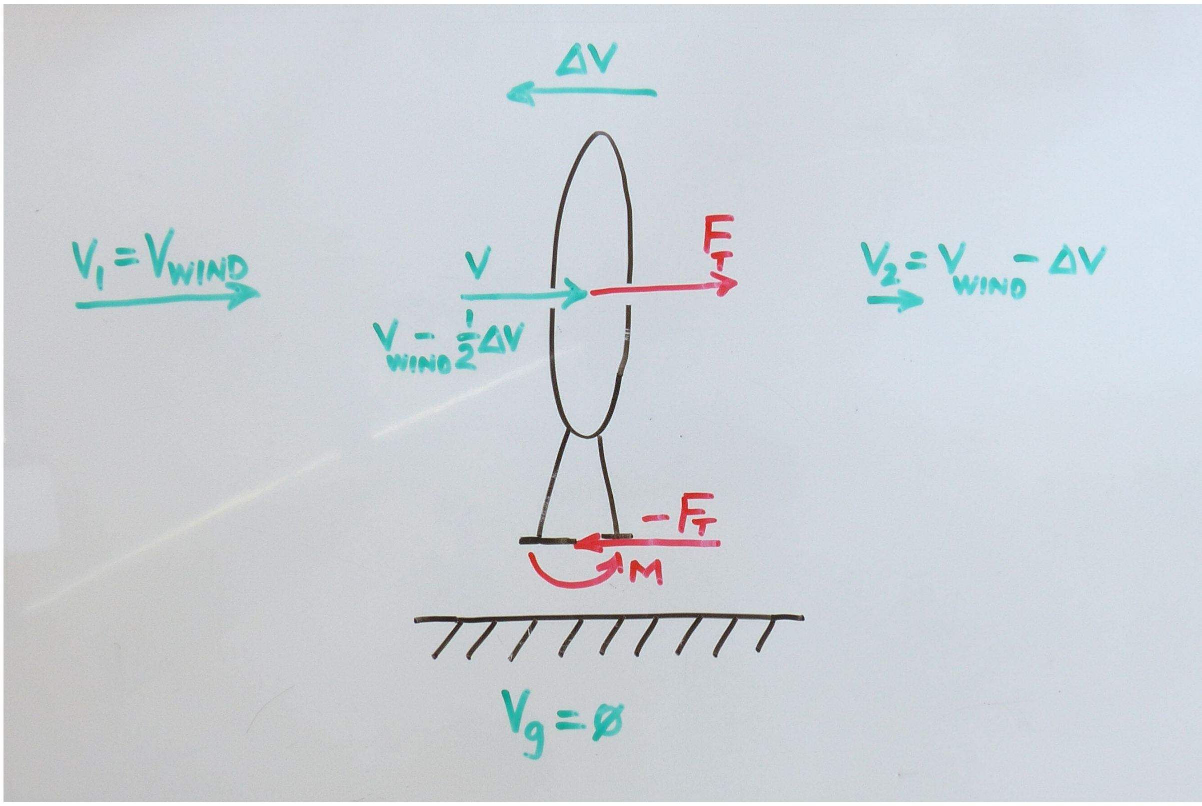

FIG. 1. Idealized stationary wind turbine (ground reference frame).

An idealized wind turbine fixed to the ground is extracting constant power, P_{net} (W), from a steady and uniform wind having constant speed equal to V_{wind} (m/s). The density of the air is ρ (kg/m^3), and the swept area of the rotor disk is S (m^2). The turbine slows the airflow down by an amount equal to ΔV (m/s), such that the air speed far downstream is reduced to $V_{\text{wind}} - \Delta V$. This negative acceleration of the airflow results in a thrust force of the air acting on the rotor in the direction of the wind equal to $\vec{F}_{\text{t}}$ (N). See figure 1. Additional parameters are given as:

$$a = \frac{1}{2}\frac{\Delta V}{V_{\text{wind}}} \quad \text{; Dimensionless “Axial Induction Factor.”} \tag{1}$$

$$V = V_{\text{wind}} - \frac{1}{2}\Delta V \quad \text{; Speed of the air passing through the rotor disk } (m/s).^{\text{a}} \tag{2}$$

$$\dot{m} = \rho S V \quad \text{; Mass flow rate of air passing through the rotor disk } (kg/s). \tag{3}$$

[a] It is generally accepted that the velocity of the air as it passes through an (idealized) actuator disk, V, (for a propeller or turbine rotor,) has a magnitude halfway between the far upstream velocity, V_1, and the far downstream velocity, V_2, i.e. $V = 1/2(V_1 + V_2)$. The mathematical proof of this concept is not provided in this paper.

FIND:

Part A: Derive dimensionless equations, each written as a function of the axial induction factor a, for: 1.) the coefficient of thrust force of the air acting on the rotor disk, $CF_{\mathrm{t}} = f(a)$; and 2.) the coefficient of net power extracted from the wind, $CP_{\mathrm{net}} = f(a)$.

Part B: Determine: 1.) the maximum possible net power coefficient, CP_{netmax}; and 2.) the axial induction factor value, a_{netmax}, required to maximize the net power. The reference constants to be used for normalizing the derived equations into dimensionless form are given as:

$$P_{\mathrm{wind}} = \frac{1}{2}\rho S V_{\mathrm{wind}}^3 \quad \text{; Total kinetic power of the wind passing through area S } (W).^{\mathrm{b}} \tag{4}$$

$$F_{\mathrm{wind}} = \frac{1}{2}\rho S V_{\mathrm{wind}}^2 \quad \text{; Force which satisfies the equation: } P_{\mathrm{wind}} = F_{\mathrm{wind}} V_{\mathrm{wind}} \ (N).^{\mathrm{b}} \tag{5}$$

Assumptions:

- The air behaves as an ideal Newtonian fluid; it is incompressible and inviscid.
- The airflow is steady, laminar and adiabatic.
- The rotor acts as an ideal 100 % efficient actuator disk with zero losses.
- Losses due to swirl in the outflow and work done on the air outside the stream tube are negligible.
- The flow is considered to be axial with uniform velocity within any cross sectional area slice of the stream tube normal to the x axis.
- A 100% efficient transmission connects the rotor to a 100% efficient electrical generator that provides P_{net} power output to a connected load.

[b] V_{wind}, P_{wind} and F_{wind}, are the reference constants used for converting the dimensional equations for velocity, power and force into dimensionless form. The P_{wind}, "*Power of the wind*" constant, represents the kinetic power that a wind flowing freely through a cross sectional area S possesses, as measured in the ground reference frame. The F_{wind}, "*Force of the wind*" constant, represents the (theoretical) force that would be required to "stop the wind" and extract the full power from the wind.

SOLUTION:

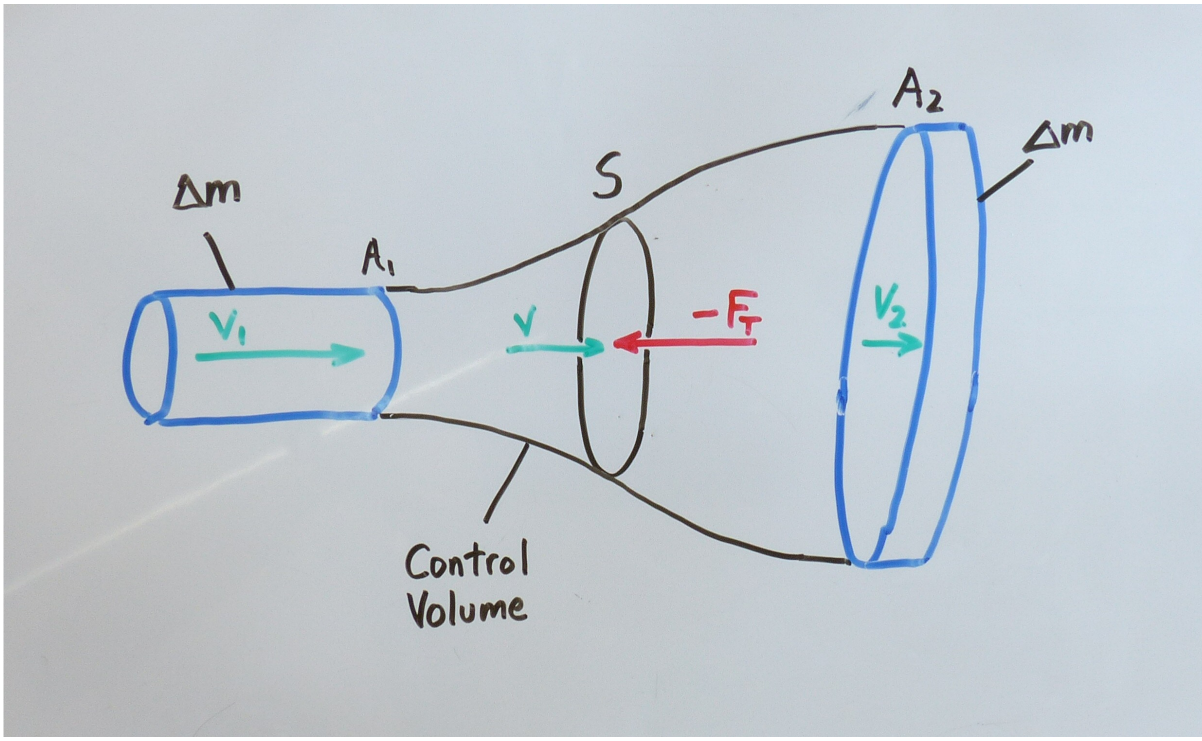

FIG. 2. Free body diagram of air passing through control volume (ground/rotor reference frame.)

In time Δt, a discreet mass of air located far upstream from the rotor disk, Δm, having constant speed, $V_1 = V_{\text{wind}}$, enters the control volume where negative acceleration is taking place. During the same Δt time period, an equal mass of air located far downstream from the rotor disk, Δm, exits the control volume with constant lesser speed, $V_2 = V_{\text{wind}} - \Delta V$. See figure 2. The effective reduction in speed of mass Δm from V_1 upstream to V_2 downstream, ΔV, is caused by the thrust reaction force of the rotor acting on the air inside the control volume, $-\vec{F}_{\text{t}}$. Applying Newton's second law, $\vec{F} = m\vec{a}$,[c] and substituting the given equation for mass flow rate, $\dot{m} = \rho SV$, we get an equation for the thrust force F_{t}:

$$-\vec{F}_{\text{t}} = \Delta m\vec{a} \quad \text{; Newton's 2nd. Substitute average acceleration, } \vec{a} = \Delta\vec{V}/\Delta t\text{:}^{\text{c}} \tag{6}$$

$$-\vec{F}_{\text{t}} = \Delta m\frac{\Delta\vec{V}}{\Delta t} \quad \text{; Both } -\vec{F}_{\text{t}} \text{ and } \Delta\vec{V} \text{ vectors have negative direction, so:} \tag{7}$$

$$-F_{\text{t}} = \Delta m(\frac{-\Delta V}{\Delta t}) \quad \text{; Converted from vector to scaler. Multiply both sides by } -1\text{:} \tag{8}$$

$$F_{\text{t}} = \Delta m\frac{\Delta V}{\Delta t} \quad \text{; Rearrange:} \tag{9}$$

$$= \frac{\Delta m}{\Delta t}\Delta V \quad \text{; Thrust force is mass flow rate } \dot{m} \text{ times } \Delta V. \tag{10}$$

$$= \dot{m}\Delta V \quad \text{; Substitute given equation, } \dot{m} = \rho SV\text{:} \tag{11}$$

$$= \rho SV\Delta V \quad \text{; Rearrange:} \tag{12}$$

$$F_{\text{t}} = \rho S\Delta VV \quad \text{; Thrust force of air acting on rotor, } F_{\text{t}} = f(V, \Delta V). \tag{13}$$

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[c] In this article, the symbol: "a" is used to represent two different parameters: 1.) the axial induction factor, and 2.) the acceleration term in Newton's second law equation, ($\vec{F} = m\vec{a}$). Hopefully, this will not cause any confusion since the acceleration term is used only briefly here.

Since all components of the turbine are 100% efficient, the net power output from the turbine, P_{net}, is equal to the power transferred from the air into the rotor, P_{ar}. Applying the general vector equation for power, $P = \vec{F} \cdot \vec{V}$, to the force acting on the air passing through the rotor disk, we get an equation for the net power:

$$P_{\text{ra}} = -\vec{F}_{\text{t}} \cdot \vec{V} \quad \text{; Power transferred from rotor into the air; vector form.} \tag{14}$$

$$P_{\text{ra}} = -F_{\text{t}} V \quad \text{; Scalar form. Negative power. But } P_{\text{ra}} = -P_{\text{ar}}\text{, so:} \tag{15}$$

$$P_{\text{ar}} = F_{\text{t}} V = P_{\text{net}} \quad \text{; Power transferred from air into the rotor. Substitute } F_{\text{t}}\text{:} \tag{16}$$

$$P_{\text{net}} = \rho S \Delta V V^2 \quad \text{; Net wind power extracted by turbine, } P_{\text{net}} = f(V, \Delta V)\text{.} \tag{17}$$

Substituting the given equation for V into the equations for F_{t} and P_{net} yields:

$$V = V_{\text{wind}} - \frac{1}{2}\Delta V \quad \text{; Given equation for } V\text{. Substitute into } F_{\text{t}} \text{ and } P_{\text{net}}\text{:} \tag{18}$$

$$F_{\text{t}} = \rho S \Delta V (V_{\text{wind}} - \frac{1}{2}\Delta V) \quad \text{; Thrust force of air on rotor, } F_{\text{t}} = f(V_{\text{wind}}, \Delta V)\text{, and} \tag{19}$$

$$P_{\text{net}} = \rho S \Delta V (V_{\text{wind}} - \frac{1}{2}\Delta V)^2 \quad \text{; Net power extracted by turbine, } P_{\text{net}} = f(V_{\text{wind}}, \Delta V)\text{.} \tag{20}$$

To normalize the equations into dimensionless form, we solve the given axial induction factor equation for ΔV, and substitute that term into the equations. First for F_{t}:

$$a = \frac{1}{2}\frac{\Delta V}{V_{\text{wind}}} \quad \text{; Given equation for } a\text{. Re-arrange:} \tag{21}$$

$$\Delta V = 2aV_{\text{wind}} \quad \text{; } \Delta V = f(V_{\text{wind}}, a)\text{. Substitute into } F_{\text{t}} \text{ equation:} \tag{22}$$

$$F_{\text{t}} = \rho S (2aV_{\text{wind}})\big(V_{\text{wind}} - \frac{1}{2}(2aV_{\text{wind}})\big) \quad \text{; } F_{\text{t}} = f(V_{\text{wind}}, a)\text{. Gather } V_{\text{wind}} \text{ terms:} \tag{23}$$

$$= \rho S V_{\text{wind}}^2 2a(1-a) \quad \text{; Cleverly multiply and divide by 2:} \tag{24}$$

$$= \frac{1}{2}\rho S V_{\text{wind}}^2 \cdot 4a(1-a) \quad \text{; Notice that first term is equal to } F_{\text{wind}}\text{, so:} \tag{25}$$

$$F_{\text{t}} = F_{\text{wind}} \cdot CF_{\text{t}} \quad \text{; Generic equation for rotor thrust force, } F_{\text{t}}\text{, where:} \tag{26}$$

$$\left[CF_{\text{t}} = 4a(1-a)\right] \quad \text{; Answer A1: } CF_{\text{t}} = f(a)\text{.} \tag{27}$$

Next substitute $\Delta V = 2aV_{\text{wind}}$ into $P_{\text{net}} = f(V_{\text{wind}}, \Delta V)$:

$$P_{\text{net}} = \rho S (2aV_{\text{wind}})(V_{\text{wind}} - \frac{1}{2}(2aV_{\text{wind}}))^2 \quad \text{; } P_{\text{net}} = f(V_{\text{wind}}, a)\text{. Gather } V_{\text{wind}} \text{ terms:} \tag{28}$$

$$= \rho S V_{\text{wind}}^3 2a(1-a)^2 \quad \text{; Multiply and divide by 2:} \tag{29}$$

$$= \frac{1}{2}\rho S V_{\text{wind}}^3 \cdot 4a(1-a)^2 \quad \text{; Notice that first term is equal to } P_{\text{wind}}\text{, so:} \tag{30}$$

$$P_{\text{net}} = P_{\text{wind}} \cdot CP_{\text{net}} \quad \text{; Generic equation for net power, } P_{\text{net}}\text{, where:} \tag{31}$$

$$\left[CP_{\text{net}} = 4a(1-a)^2\right] \quad \text{; Answer A2: } CP_{\text{net}} = f(a)\text{.} \tag{32}$$

Figure 3 shows a plot of CFt and CPnet. To determine the maximum possible net power coefficient, we take the derivative of the $CP_{\text{net}} = f(a)$ equation with respect to a, set the resulting equation equal to zero, then solve for the

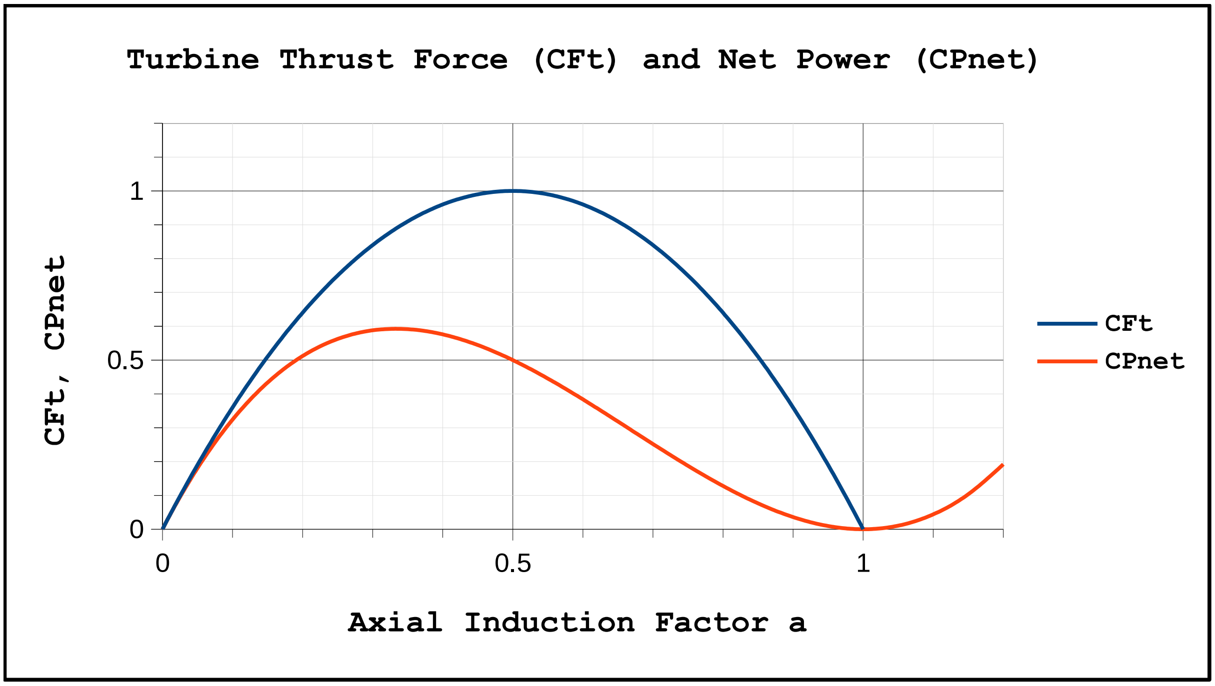

FIG. 3. Plot of functions CF_{t} and CP_{net} vs Axial Induction Factor a.

roots:

$$CP_{\mathrm{net}} = 4a(1-a)^2 \quad ;\ CP_{\mathrm{net}} = f(a).\ \text{First expand into a polynomial:} \tag{33}$$

$$= 4a(1-a)(1-a) \quad ; \tag{34}$$

$$= 4a(1-2a+a^2) \quad ; \tag{35}$$

$$CP_{\mathrm{net}} = 4a^3 - 8a^2 + 4a \quad ;\ CP_{\mathrm{net}}\ \text{in polynomial form. Take the derivative:} \tag{36}$$

$$\frac{d(CP_{\mathrm{net}})}{da} = 12a^2 - 16a + 4 \quad ;\ \text{Derivative of } CP_{\mathrm{net}}.\ \text{Set equal to 0 and divide by 4:} \tag{37}$$

$$0 = 3a^2 - 4a + 1 \quad ;\ \text{Roots of this equation are } a_{\mathrm{netmax}} \text{ and } a_{\mathrm{netmin}} \text{ of } CP_{\mathrm{net}}. \tag{38}$$

$$roots = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \quad ;\ \text{Quadratic equation. Use: } A = 3,\ B = -4 \text{ and } C = 1: \tag{39}$$

$$= \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(1)}}{2(3)} \tag{40}$$

$$= \frac{4 \pm \sqrt{16 - 12}}{6} \quad ; \tag{41}$$

$$= \frac{1}{6}(4 \pm 2) \quad ; \tag{42}$$

$$= \frac{1}{3}(2 \pm 1) \quad ; \tag{43}$$

$$roots = 1, \frac{1}{3} \quad ;\ \text{Root values: } 1 = a_{\mathrm{netmin}} \text{ and } 1/3 = a_{\mathrm{netmax}}. \tag{44}$$

$$\left[a_{\mathrm{netmax}} = \frac{1}{3}\right] \quad ;\ \text{Answer B2: Value of } a \text{ which maximizes } CP_{\mathrm{net}}. \tag{45}$$

Plugging a_{netmax} into $CP_{\text{net}} = f(a)$ reveals the maximum possible net power coefficient:

$$CP_{\text{net}} = 4a(1-a)^2 \qquad ; CP_{\text{net}} = f(a).\ \text{Plug in } a_{\text{netmax}} = 1/3: \tag{46}$$

$$CP_{\text{netmax}} = 4\left(\frac{1}{3}\right)\left(1 - \frac{1}{3}\right)^2 \qquad ; \tag{47}$$

$$= \frac{4}{3}\left(\frac{2}{3}\right)^2 \qquad ; \tag{48}$$

$$= \frac{4}{3}\left(\frac{4}{9}\right) \qquad ; \tag{49}$$

$$\left[CP_{\text{netmax}} = \frac{16}{27}\right] \qquad ; \text{Answer B1: } CP_{\text{netmax}} = 16/27 = \text{Betz Limit.} \tag{50}$$