1 Nomenclature for the Generalized DDW & DUW Cart Equations: Rev:20240107_0300

Table 1: Given constants (dimensional).

Constant	Equation	Description	Units
ρ		Density of the fluid (air).	kg/m^3
S		Swept cross sectional area of rotor disk.	m^2
$V_{\text{wind}} = V_{\text{w}}$		Velocity of the wind relative to the ground.[a]	m/s
$P_{\text{wind}} =$	$\frac{1}{2}\rho S V_{\text{wind}}^3$	Kinetic power of wind freely passing through area S.[a]	W
$F_{\text{wind}} =$	$\frac{1}{2}\rho S V_{\text{wind}}^2$	Force which satisfies equation: $P_{\text{wind}} = F_{\text{wind}} V_{\text{wind}}$.	N

Table 2: Independent variables (dimensional).

Variable	Description	Units
$V_{\text{cart}} = V_{\text{c}}$	Velocity of the cart, (either DUW or DDW).[a]	m/s
ΔV	Total change of linear air velocity caused by rotor.[c]	m/s

Table 3: Independent variables (dimensionless).

Variable	Equation	Description
$n =$	$\frac{V_{\text{cart}}}{V_{\text{wind}}}$	Dimensionless velocity of the cart, (either DUW or DDW).[a]
$\Delta =$	$\frac{\Delta V}{V_{\text{wind}}}$	Dimensionless change of air velocity caused by rotor.[c]
$a =$	$\frac{1}{2}\Delta = \frac{1}{2}\frac{\Delta V}{V_{\text{wind}}}$	Dimensionless "Axial Induction Factor."[c]

Table 4: Dependent parameters (dimensional).

Parameter	Description	Dependency	Units
F_{t}	Thrust force of the air acting on the rotor in the direction of the wind.	$f(V_{\text{c}}, \Delta V)$	N
P_{ra}	Power transferred from the (propeller) rotor into the air.[b]	$f(V_{\text{c}}, \Delta V)$	W
P_{ar}	Power transferred from the air into the (turbine) rotor.[b]	$f(V_{\text{c}}, \Delta V)$	W
F_{g}	Ground force acting on the wheels in the direction opposite of the wind.	$f(V_{\text{c}}, \Delta V)$	N
F_{net}	Net forward force in the direction of the cart velocity ($\vec{F}_{\text{net}} = \vec{F}_{\text{t}} + \vec{F}_{\text{g}}$).	$f(V_{\text{c}}, \Delta V)$	N
P_{net}	Net power extracted from the wind by the cart ($P_{\text{net}} = \vec{F}_{\text{net}} \cdot \vec{V}_{\text{cart}}$).[a]	$f(V_{\text{c}}, \Delta V)$	W
ΔV_{netmax}	ΔV value which maximizes both F_{net} and P_{net} for given cart speed.	$f(V_{\text{c}})$	m/s
P_{netmax}	Maximum net power extracted from the wind for given cart speed.[ad]	$f(V_{\text{c}})$	W

[a] Ground frame of reference. The propeller cart travels directly downwind (DDW,) in the same direction as the wind, at a speed greater than or equal to the wind. The turbine cart travels directly upwind (DUW,) in the direction opposite of the wind, at a speed greater than or equal to zero.

[b] Cart frame of reference.

[c] Any frame of reference. Note that the ΔV velocity vector, which is the total change in linear velocity of the air in the stream tube caused by the rotor, always points in the direction opposite of the wind. This is true for both the DDW propeller cart and the DUW turbine cart.

[d] Hats off to Brother Daniel who had the unmitigated audacity (and courage) to actually attempt to substitute the equation for the optimal $\Delta V_{\text{netmax}} = f(V_{\text{c}})$ into the equation for $P_{\text{net}} = f(V_{\text{c}}, \Delta V)$, to come up with this *one equation to rule them all*, $P_{\text{netmax}} = f(V_{\text{c}})$, which defines the maximum extractible net power as a function of just one variable, the cart speed. His pioneering work on this subject provided this author the inspiration to pursue this derivation for the past two plus years. *Thanks Dan!*

2 Directly Downwind Idealized Propeller Cart Equations: $(V_{\textbf{cart}} \geq V_{\textbf{wind}})$

Table 5: Dimensional DDW idealized propeller cart equations: $f(V_{\mathrm{cart}}, \Delta V)$.

Parameter	$f(V_\mathrm{c}, \Delta V)$	$f(V_\mathrm{c} = V_\mathrm{w}, \Delta V)^\mathrm{e}$	$f(V_\mathrm{c}, \Delta V = V_\mathrm{w})^\mathrm{f}$
$F_\mathrm{t} =$	$\rho S \Delta V (V_\mathrm{c} - V_\mathrm{w} + \frac{1}{2}\Delta V)$	$\frac{1}{2}\rho S \Delta V^2$	$\rho S V_\mathrm{w}(V_\mathrm{c} - \frac{1}{2}V_\mathrm{w})$
$P_\mathrm{ra} =$	$\rho S \Delta V (V_\mathrm{c} - V_\mathrm{w} + \frac{1}{2}\Delta V)^2$	$\frac{1}{4}\rho S \Delta V^3$	$\rho S V_\mathrm{w}(V_\mathrm{c} - \frac{1}{2}V_\mathrm{w})^2$
$F_\mathrm{g} =$	$\frac{\rho S \Delta V}{V_\mathrm{c}}(V_\mathrm{c} - V_\mathrm{w} + \frac{1}{2}\Delta V)^2$	$\frac{1}{4}\frac{\rho S \Delta V^3}{V_\mathrm{w}}$	$\frac{\rho S V_\mathrm{w}}{V_\mathrm{c}}(V_\mathrm{c} - \frac{1}{2}V_\mathrm{w})^2$
$F_\mathrm{net} =$	$\frac{\rho S}{V_\mathrm{c}}\big(-\frac{1}{4}\Delta V^3 + (V_\mathrm{w} - \frac{1}{2}V_\mathrm{c})\Delta V^2 + (V_\mathrm{c}V_\mathrm{w} - V_\mathrm{w}^2)\Delta V\big)$	$\frac{1}{2}\frac{\rho S}{V_\mathrm{w}}(V_\mathrm{w}\Delta V^2 - \frac{1}{2}\Delta V^3)$	$\frac{1}{2}\frac{\rho S V_\mathrm{w}^2}{V_\mathrm{c}}(V_\mathrm{c} - \frac{1}{2}V_\mathrm{w})$
$P_\mathrm{net} =$	$\rho S\big(-\frac{1}{4}\Delta V^3 + (V_\mathrm{w} - \frac{1}{2}V_\mathrm{c})\Delta V^2 + (V_\mathrm{c}V_\mathrm{w} - V_\mathrm{w}^2)\Delta V\big)$	$\frac{1}{2}\rho S(V_\mathrm{w}\Delta V^2 - \frac{1}{2}\Delta V^3)$	$\frac{1}{2}\rho S V_\mathrm{w}^2(V_\mathrm{c} - \frac{1}{2}V_\mathrm{w})$
$\Delta V_\mathrm{netmax} =$	$\frac{2}{3}\big(2V_\mathrm{w} - V_\mathrm{c} + \sqrt{V_\mathrm{c}^2 - V_\mathrm{c}V_\mathrm{w} + V_\mathrm{w}^2}\big)$	$\frac{4}{3}V_\mathrm{w}$	
$P_\mathrm{netmax} =$	$\frac{2}{27}\rho S\big((V_\mathrm{c} + V_\mathrm{w})^3 - 3(V_\mathrm{c}^3 + V_\mathrm{w}^3) + 2(V_\mathrm{c}^2 - V_\mathrm{c}V_\mathrm{w} + V_\mathrm{w}^2)^{\frac{3}{2}}\big)$	$\frac{16}{27}P_\mathrm{wind}$	

Table 6: Dimensionless DDW idealized propeller cart equations: $f(n, \Delta)$.

Parameter	$f(n, \Delta)$	$f(n = 1, \Delta)^\mathrm{e}$	$f(n, \Delta = 1)^\mathrm{f}$
$CF_\mathrm{t} = \frac{F_\mathrm{t}}{F_\mathrm{wind}} =$	$2\Delta(n - 1 + \frac{1}{2}\Delta)$	Δ^2	$2(n - \frac{1}{2})$
$CP_\mathrm{ra} = \frac{P_\mathrm{ra}}{P_\mathrm{wind}} =$	$2\Delta(n - 1 + \frac{1}{2}\Delta)^2$	$\frac{1}{2}\Delta^3$	$2(n - \frac{1}{2})^2$
$CF_\mathrm{g} = \frac{F_\mathrm{g}}{F_\mathrm{wind}} =$	$\frac{2\Delta}{n}(n - 1 + \frac{1}{2}\Delta)^2$	$\frac{1}{2}\Delta^3$	$\frac{2}{n}(n - \frac{1}{2})^2$
$CF_\mathrm{net} = \frac{F_\mathrm{net}}{F_\mathrm{wind}} =$	$\frac{1}{n}\big(-\frac{1}{2}\Delta^3 + (2 - n)\Delta^2 + (2n - 2)\Delta\big)$	$\Delta^2 - \frac{1}{2}\Delta^3$	$\frac{1}{n}(n - \frac{1}{2})$
$CP_\mathrm{net} = \frac{P_\mathrm{net}}{P_\mathrm{wind}} =$	$-\frac{1}{2}\Delta^3 + (2 - n)\Delta^2 + (2n - 2)\Delta$	$\Delta^2 - \frac{1}{2}\Delta^3$	$n - \frac{1}{2}$
$\Delta_\mathrm{netmax} = \frac{\Delta V_\mathrm{netmax}}{V_\mathrm{wind}} =$	$\frac{2}{3}(2 - n + \sqrt{n^2 - n + 1})$	$\frac{4}{3}$	
$CP_\mathrm{netmax} = \frac{P_\mathrm{netmax}}{P_\mathrm{wind}} =$	$\frac{4}{27}\big((n + 1)^3 - 3(n^3 + 1) + 2(n^2 - n + 1)^{\frac{3}{2}}\big)$	$\frac{16}{27}$	

Table 7: Dimensionless DDW idealized propeller cart equations: $f(n, a)$.

Parameter	$f(n, a)$	$f(n = 1, a)^\mathrm{e}$	$f(n, a = \frac{1}{2})^\mathrm{f}$
$CF_\mathrm{t} = \frac{F_\mathrm{t}}{F_\mathrm{wind}} =$	$4a(n - 1 + a)$	$4a^2$	$2(n - \frac{1}{2})$
$CP_\mathrm{ra} = \frac{P_\mathrm{ra}}{P_\mathrm{wind}} =$	$4a(n - 1 + a)^2$	$4a^3$	$2(n - \frac{1}{2})^2$
$CF_\mathrm{g} = \frac{F_\mathrm{g}}{F_\mathrm{wind}} =$	$\frac{4a}{n}(n - 1 + a)^2$	$4a^3$	$\frac{2}{n}(n - \frac{1}{2})^2$
$CF_\mathrm{net} = \frac{F_\mathrm{net}}{F_\mathrm{wind}} =$	$\frac{4}{n}\big(-a^3 + (2 - n)a^2 + (n - 1)a\big)$	$4a^2 - 4a^3$	$\frac{1}{n}(n - \frac{1}{2})$
$CP_\mathrm{net} = \frac{P_\mathrm{net}}{P_\mathrm{wind}} =$	$4\big(-a^3 + (2 - n)a^2 + (n - 1)a\big)$	$4a^2 - 4a^3$	$n - \frac{1}{2}$
$a_\mathrm{netmax} = \frac{V_\mathrm{netmax}}{V_\mathrm{wind}} =$	$\frac{1}{3}(2 - n + \sqrt{n^2 - n + 1})$	$\frac{2}{3}$	
$CP_\mathrm{netmax} = \frac{P_\mathrm{netmax}}{P_\mathrm{wind}} =$	$\frac{4}{27}\big((n + 1)^3 - 3(n^3 + 1) + 2(n^2 - n + 1)^{\frac{3}{2}}\big)$	$\frac{16}{27}$	

[e] The equations in the third column represent the special case where the propeller cart speed is held constant and equal to the speed of the wind, i.e. where: $V_\mathrm{cart} = V_\mathrm{wind}$, (and $n = 1$.) This represents the special case for a cart hovering on a treadmill.

[f] The equations in the fourth column represent the special case where the ΔV term is held constant and equal to the speed of the wind, i.e. where: $\Delta V = V_\mathrm{wind}$, (and $\Delta = 1$ and $a = \frac{1}{2}$). Note that for this special case, the net power coefficient equation for the propeller cart simplifies to the straight line: $CP_\mathrm{net} = n - \frac{1}{2}$.

3 Directly Upwind Idealized Turbine Cart Equations: ($V_{\mathbf{cart}} \geq 0$)

Table 8: Dimensional DUW idealized turbine cart equations: $f(V_{\mathrm{cart}}, \Delta V)$.

Parameter	$f(V_{\mathrm{c}}, \Delta V)$	$f(V_{\mathrm{c}} = 0, \Delta V)$[g]	$f(V_{\mathrm{c}}, \Delta V = V_{\mathrm{w}})$[h]
$F_{\mathrm{t}} =$	$\rho S \Delta V(V_{\mathrm{c}} + V_{\mathrm{w}} - \frac{1}{2}\Delta V)$	$\rho S \Delta V(V_{\mathrm{w}} - \frac{1}{2}\Delta V)$	$\rho S V_{\mathrm{w}}(V_{\mathrm{c}} + \frac{1}{2}V_{\mathrm{w}})$
$P_{\mathrm{ar}} =$	$\rho S \Delta V(V_{\mathrm{c}} + V_{\mathrm{w}} - \frac{1}{2}\Delta V)^2$	$\rho S \Delta V(V_{\mathrm{w}} - \frac{1}{2}\Delta V)^2$	$\rho S V_{\mathrm{w}}(V_{\mathrm{c}} + \frac{1}{2}V_{\mathrm{w}})^2$
$F_{\mathrm{g}} =$	$\frac{\rho S \Delta V}{V_{\mathrm{c}}}(V_{\mathrm{c}} + V_{\mathrm{w}} - \frac{1}{2}\Delta V)^2$	Undefined[i] ∞	$\frac{\rho S V_{\mathrm{w}}}{V_{\mathrm{c}}}(V_{\mathrm{c}} + \frac{1}{2}V_{\mathrm{w}})^2$
$F_{\mathrm{net}} =$	$\frac{\rho S}{V_{\mathrm{c}}}\big(\frac{1}{4}\Delta V^3 - (V_{\mathrm{w}} + \frac{1}{2}V_{\mathrm{c}})\Delta V^2 + (V_{\mathrm{c}}V_{\mathrm{w}} + V_{\mathrm{w}}^2)\Delta V\big)$	Undefined[i] ∞	$\frac{1}{2}\frac{\rho S V_{\mathrm{w}}^2}{V_{\mathrm{c}}}(V_{\mathrm{c}} + \frac{1}{2}V_{\mathrm{w}})$
$P_{\mathrm{net}} =$	$\rho S\big(\frac{1}{4}\Delta V^3 - (V_{\mathrm{w}} + \frac{1}{2}V_{\mathrm{c}})\Delta V^2 + (V_{\mathrm{c}}V_{\mathrm{w}} + V_{\mathrm{w}}^2)\Delta V\big)$	$\rho S \Delta V(V_{\mathrm{w}} - \frac{1}{2}\Delta V)^2$	$\frac{1}{2}\rho S V_{\mathrm{w}}^2(V_{\mathrm{c}} + \frac{1}{2}V_{\mathrm{w}})$
$\Delta V_{\mathrm{netmax}} =$	$\frac{2}{3}\big(2V_{\mathrm{w}} + V_{\mathrm{c}} - \sqrt{V_{\mathrm{c}}^2 + V_{\mathrm{c}}V_{\mathrm{w}} + V_{\mathrm{w}}^2}\big)$	$\frac{2}{3}V_{\mathrm{w}}$	
$P_{\mathrm{netmax}} =$	$\frac{2}{27}\rho S\big((V_{\mathrm{c}} - V_{\mathrm{w}})^3 - 3(V_{\mathrm{c}}^3 - V_{\mathrm{w}}^3) + 2(V_{\mathrm{c}}^2 + V_{\mathrm{c}}V_{\mathrm{w}} + V_{\mathrm{w}}^2)^{\frac{3}{2}}\big)$	$\frac{16}{27}P_{\mathrm{wind}}$	

Table 9: Dimensionless DUW idealized turbine cart equations: $f(n, \Delta)$.

Parameter	$f(n, \Delta)$	$f(n = 0, \Delta)$[g]	$f(n, \Delta = 1)$[h]
$CF_{\mathrm{t}} = \frac{F_{\mathrm{t}}}{F_{\mathrm{wind}}} =$	$2\Delta(n + 1 - \frac{1}{2}\Delta)$	$2\Delta(1 - \frac{1}{2}\Delta)$	$2(n + \frac{1}{2})$
$CP_{\mathrm{ar}} = \frac{P_{\mathrm{ar}}}{P_{\mathrm{wind}}} =$	$2\Delta(n + 1 - \frac{1}{2}\Delta)^2$	$2\Delta(1 - \frac{1}{2}\Delta)^2$	$2(n + \frac{1}{2})^2$
$CF_{\mathrm{g}} = \frac{F_{\mathrm{g}}}{F_{\mathrm{wind}}} =$	$\frac{2\Delta}{n}(n + 1 - \frac{1}{2}\Delta)^2$	Undefined[i] ∞	$\frac{2}{n}(n + \frac{1}{2})^2$
$CF_{\mathrm{net}} = \frac{F_{\mathrm{net}}}{F_{\mathrm{wind}}} =$	$\frac{1}{n}\big(\frac{1}{2}\Delta^3 - (2 + n)\Delta^2 + (2n + 2)\Delta\big)$	Undefined[i] ∞	$\frac{1}{n}(n + \frac{1}{2})$
$CP_{\mathrm{net}} = \frac{P_{\mathrm{net}}}{P_{\mathrm{wind}}} =$	$\frac{1}{2}\Delta^3 - (2 + n)\Delta^2 + (2n + 2)\Delta$	$2\Delta(1 - \frac{1}{2}\Delta)^2$	$n + \frac{1}{2}$
$\Delta_{\mathrm{netmax}} = \frac{\Delta V_{\mathrm{netmax}}}{V_{\mathrm{wind}}} =$	$\frac{2}{3}(2 + n - \sqrt{n^2 + n + 1})$	$\frac{2}{3}$	
$CP_{\mathrm{netmax}} = \frac{P_{\mathrm{netmax}}}{P_{\mathrm{wind}}} =$	$\frac{4}{27}\big((n - 1)^3 - 3(n^3 - 1) + 2(n^2 + n + 1)^{\frac{3}{2}}\big)$	$\frac{16}{27}$	

Table 10: Dimensionless DUW idealized turbine cart equations: $f(n, a)$.

Parameter	$f(n, a)$	$f(n = 0, a)$[g]	$f(n, a = \frac{1}{2})$[h]
$CF_{\mathrm{t}} = \frac{F_{\mathrm{t}}}{F_{\mathrm{wind}}} =$	$4a(n + 1 - a)$	$4a(1 - a)$	$2(n + \frac{1}{2})$
$CP_{\mathrm{ar}} = \frac{P_{\mathrm{ar}}}{P_{\mathrm{wind}}} =$	$4a(n + 1 - a)^2$	$4a(1 - a)^2$	$2(n + \frac{1}{2})^2$
$CF_{\mathrm{g}} = \frac{F_{\mathrm{g}}}{F_{\mathrm{wind}}} =$	$\frac{4a}{n}(n + 1 - a)^2$	Undefined[i] ∞	$\frac{2}{n}(n + \frac{1}{2})^2$
$CF_{\mathrm{net}} = \frac{F_{\mathrm{net}}}{F_{\mathrm{wind}}} =$	$\frac{4}{n}\big(a^3 - (2 + n)a^2 + (n + 1)a\big)$	Undefined[i] ∞	$\frac{1}{n}(n + \frac{1}{2})$
$CP_{\mathrm{net}} = \frac{P_{\mathrm{net}}}{P_{\mathrm{wind}}} =$	$4\big(a^3 - (2 + n)a^2 + (n + 1)a\big)$	$4a(1 - a)^2$	$n + \frac{1}{2}$
$a_{\mathrm{netmax}} = \frac{V_{\mathrm{netmax}}}{V_{\mathrm{wind}}} =$	$\frac{1}{3}(2 + n - \sqrt{n^2 + n + 1})$	$\frac{1}{3}$	
$CP_{\mathrm{netmax}} = \frac{P_{\mathrm{netmax}}}{P_{\mathrm{wind}}} =$	$\frac{4}{27}\big((n - 1)^3 - 3(n^3 - 1) + 2(n^2 + n + 1)^{\frac{3}{2}}\big)$	$\frac{16}{27}$	

[g] The equations in the third column represent the special case where the turbine cart is stopped, i.e. where: $V_{\mathrm{cart}} = 0$, (and $n = 0$.) This represents the special case for the cart operating as a stationary wind turbine.

[h] The equations in the fourth column represent the special case where the ΔV term is held constant and equal to the speed of the wind, i.e. where: $\Delta V = V_{\mathrm{wind}}$, (and $\Delta = 1$ and $a = \frac{1}{2}$). Note that for this special case, the net power coefficient equation for the turbine cart simplifies to the straight line: $CP_{\mathrm{net}} = n + \frac{1}{2}$.

[i] For the special case where the turbine cart is stopped, useful power can still be harvested at the rotor, but the wheels are not moving so the ground force on the wheels, F_{g}, would need to be infinitely large to absorb this power. And since the net forward force is the ground force minus the thrust force, i.e. $F_{\mathrm{net}} = F_{\mathrm{g}} - F_{\mathrm{t}}$, it too, would be infinite. Since these values are clearly nonsensical, the equations for F_{g} and F_{net} are undefined for zero cart speed. Thus, for this special case where the cart is stopped, the power from the rotor is disconnected from the wheels, (using a clutch and transmission,) and connected directly to the electrical generator, and parking brakes are applied to prevent cart motion. So when the ideal Blackbird starts off on its upwind run, the brakes are released, the transmission is engaged, the clutch is popped, and Spork (or JB!) proceeds to burn rubber and pop a wheelie off into the sunset!