Spork Limit Derivation
Rev:20240303_1800

Jeffrey M. Roberson
(Dated: March 3, 2024)

GIVEN:

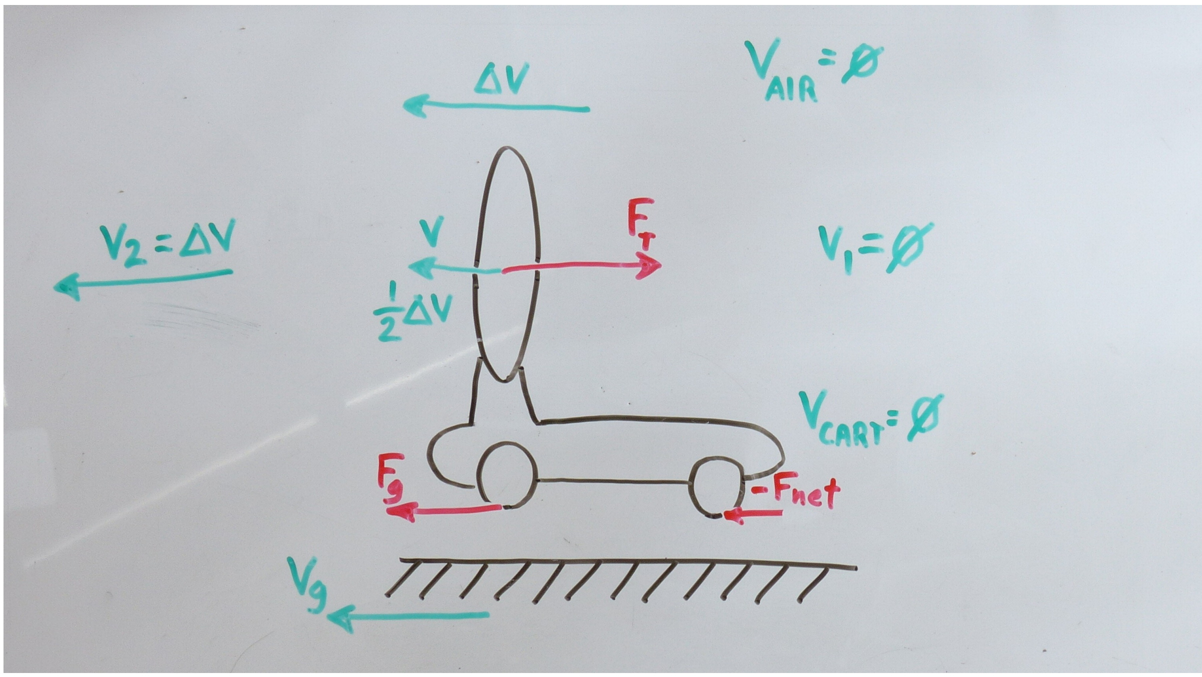

FIG. 1. Free body diagram of DDW idealized propeller cart traveling at wind speed (cart reference frame).

An idealized wind powered propeller cart is traveling directly downwind at a steady speed equal to $V_{\text{cart}} = V_{\text{wind}}$ (m/s).[a] The density of the air is ρ (kg/m^3), and the swept area of the propeller rotor disk is S (m^2). From the reference frame of the cart, the propeller, which is operating in a "*Static Thrust*" condition, is continuously taking in stationary air from far in front of the rotor and speeding it up by an amount equal to ΔV (m/s), such that the air speed far downstream is equal to ΔV, in the rearward direction.[b] This rearward acceleration of the air results in a forward thrust force of the air acting on the rotor equal to $\vec{F_{\text{t}}}$ (N). See figure 1.

An ideal transmission directly connects the propeller to the rear wheels of the cart. The ground, which in the cart frame is moving rearwards at the speed of the wind,[c] applies a rearward braking force to the bottom of the rolling rear wheels, $\vec{F_{\text{g}}}$ (N), and this force provides the necessary power to drive the propeller. When the magnitude of the forward thrust force on the propeller, $\vec{F_{\text{t}}}$, is greater than the rearward braking force on the rear wheels, $\vec{F_{\text{g}}}$, a net forward thrust force exists having a magnitude equal to $F_{\text{net}} = F_{\text{t}} - F_{\text{g}}$ (N). To counteract this net forward force and achieve steady state force equilibrium, the cart is fitted with a regenerative braking system connected to the front wheels, which is controlled by the pilot to maintain constant speed. Application of these brakes results in an equal and opposite force, $-\vec{F}_{\text{net}}$, at the bottom of the front wheels, and this force drives an ideal electrical generator which

[a] Both $\vec{V_{\text{cart}}}$ and $\vec{V_{\text{wind}}}$ velocity vectors are defined in the ground reference frame and point to the right in the plus x-axis direction. From here forward, they will simply be referred to using their (positive) scalar values. All other velocity vectors are defined in the cart reference frame and point in the directions shown in the figures.

[b] The x-axis is oriented pointing to the right, in the same direction as the wind, so the "rearward direction" of the ΔV velocity vector, refers to the minus x direction, to the left.

harvests the net power extracted from the wind-ground interface, P_{net} (W). Additional parameters are given as:

$$a = \frac{1}{2}\frac{\Delta V}{V_{\text{wind}}} \quad \text{; Dimensionless “Axial Induction Factor.”} \tag{1}$$

$$V = \frac{1}{2}\Delta V \quad \text{; Speed of the air passing through the rotor disk } (m/s).^{\text{d}} \tag{2}$$

$$\dot{m} = \rho S V \quad \text{; Mass flow rate of air passing through the rotor disk } (kg/s). \tag{3}$$

FIND:

Part A: From the cart reference frame, derive dimensionless equations, each written as a function of the axial induction factor a, for: 1.) the coefficient of forward thrust force of the air acting on the rotor disk, $CF_{\text{t}} = f(a)$; 2.) the coefficient of power transferred from the rotor into the air, $CP_{\text{ra}} = f(a)$; 3.) the coefficient of the rearward braking force of the ground acting on the rear wheels, $CF_{\text{g}} = f(a)$; 4.) the coefficient of the net forward force, $CF_{\text{net}} = f(a)$; and 5.) the coefficient of net power extracted from the wind, $CP_{\text{net}} = f(a)$.

Part B: Determine: 1.) the maximum possible net power coefficient, CP_{netmax}; and 2.) the axial induction factor value, a_{netmax}, required to maximize the net power. The reference constants to be used for normalizing the derived equations into dimensionless form are given as:

$$P_{\text{wind}} = \frac{1}{2}\rho S V_{\text{wind}}^3 \quad \text{; Total kinetic power of the wind passing through area S } (W).^{\text{e}} \tag{4}$$

$$F_{\text{wind}} = \frac{1}{2}\rho S V_{\text{wind}}^2 \quad \text{; Force which satisfies the equation: } P_{\text{wind}} = F_{\text{wind}} V_{\text{wind}} \ (N).^{\text{e}} \tag{5}$$

Assumptions:

- The air behaves as an ideal Newtonian fluid; it is incompressible and inviscid.
- The airflow is steady, laminar and adiabatic.
- The rotor acts as an ideal 100 % efficient actuator disk with zero aerodynamic losses.
- Losses due to swirl in the outflow and work done on the air outside the stream tube are negligible.
- The flow is considered to be axial with uniform velocity within any cross sectional area slice of the stream tube normal to the x axis.
- All wheels have zero rolling resistance and a static coefficient of friction adequate to prevent skidding.
- A 100% efficient transmission connects the rear wheels to the propeller rotor.
- A 100% efficient transmission connects the front wheels to a 100% efficient electrical generator that provides P_{net} power output to a connected load.

SOLUTION:

In time Δt, a discreet mass of air located far upstream from the rotor disk, Δm, having constant speed, $V_1 = 0$, enters the control volume where rearward acceleration is taking place. During the same Δt time period, an equal mass of air located far downstream from the rotor disk, Δm, exits the control volume with constant greater speed,

---

[c] The velocity of the ground in the cart reference frame has the same magnitude, but opposite direction, as the velocities of both the wind and cart in the ground reference frame. In this article, vector quantities are indicated with an arrow above the symbol. When written in scalar form, the arrow is not indicated. So the following statements are all true: $-\vec{V}_{\text{g}} = \vec{V}_{\text{wind}}$, $|\vec{V}_{\text{g}}| = |\vec{V}_{\text{wind}}|$, and $V_{\text{g}} = V_{\text{wind}}$.

[d] It is generally accepted that the velocity of the air as it passes through an (idealized) actuator disk, V, (for a propeller or turbine rotor,) has a magnitude halfway between the far upstream velocity, V_1, and the far downstream velocity, V_2, i.e. $V = 1/2(V_1 + V_2)$. The mathematical proof of this concept is not provided in this paper.

[e] V_{wind}, P_{wind} and F_{wind}, are the reference constants used for converting the dimensional equations for velocity, power and force into dimensionless form. The P_{wind}, “*Power of the wind*” constant, represents the kinetic power that a wind flowing freely through a cross sectional area S possesses, as measured in the ground reference frame. The F_{wind}, “*Force of the wind*” constant, represents the (theoretical) force that would be required to “stop the wind” and extract the full power from the wind.

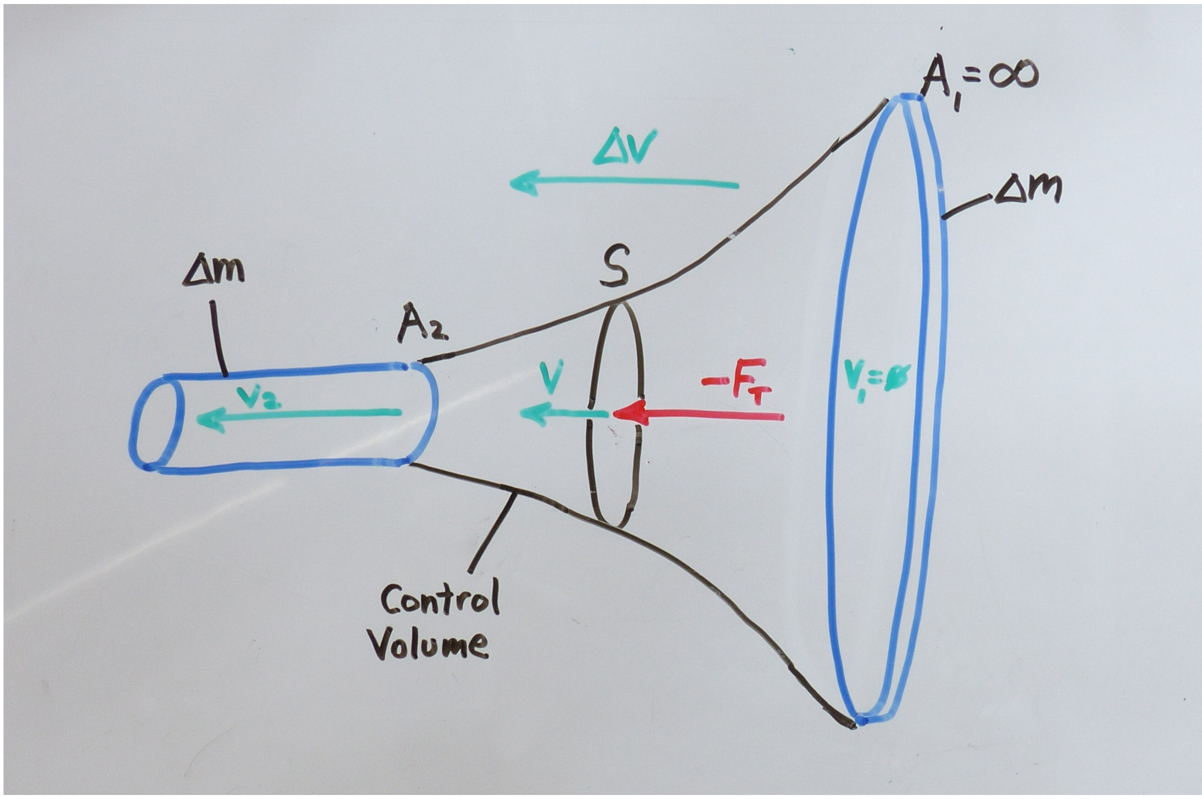

FIG. 2. Free body diagram of air passing through control volume (cart reference frame.)

$V_2 = \Delta V$. See figure 2. The effective increase in speed of mass Δm from V_1 upstream to V_2 downstream, ΔV, is caused by the thrust reaction force of the rotor acting on the air inside the control volume, $-\vec{F}_\mathrm{t}$. Applying Newton's second law, $\vec{F} = m\vec{a}$,[f] and substituting the given equation for mass flow rate, $\dot{m} = \rho S V$, we get an equation for the thrust force, F_t:

$$-\vec{F}_\mathrm{t} = \Delta m \vec{a} \qquad \text{; Newton's 2nd. Substitute average acceleration, } \vec{a} = \Delta\vec{V}/\Delta t\text{:}^{\mathrm{f}} \tag{6}$$

$$-\vec{F}_\mathrm{t} = \Delta m \frac{\Delta\vec{V}}{\Delta t} \qquad \text{; Both } -\vec{F}_\mathrm{t} \text{ and } \Delta\vec{V} \text{ vectors have negative direction, so:} \tag{7}$$

$$-F_\mathrm{t} = \Delta m (\frac{-\Delta V}{\Delta t}) \qquad \text{; Converted from vector to scaler. Multiply both sides by } -1\text{:} \tag{8}$$

$$F_\mathrm{t} = \Delta m \frac{\Delta V}{\Delta t} \qquad \text{; Rearrange:} \tag{9}$$

$$= \frac{\Delta m}{\Delta t} \Delta V \qquad \text{; Thrust force is mass flow rate } \dot{m} \text{ times } \Delta V. \tag{10}$$

$$= \dot{m} \Delta V \qquad \text{; Substitute given equation, } \dot{m} = \rho S V\text{:} \tag{11}$$

$$= \rho S V \Delta V \qquad \text{; Rearrange:} \tag{12}$$

$$F_\mathrm{t} = \rho S \Delta V V \qquad \text{; Thrust force of air acting on rotor, } F_\mathrm{t} = f(V, \Delta V). \tag{13}$$

Applying the general vector equation for power, $P = \vec{F} \cdot \vec{V}$, to the thrust reaction force acting on the air passing

---

[f] In this article, the symbol: "a" is used to represent two different parameters: 1.) the axial induction factor, and 2.) the acceleration term in Newton's second law equation, ($\vec{F} = m\vec{a}$). Hopefully, this will not cause any confusion since the acceleration term is used only briefly here.

through the rotor disk, we get an equation for the power transferred from the rotor into the air, P_{ra}:

$$P_{\mathrm{ra}} = -\vec{F}_{\mathrm{t}} \cdot \vec{V} \quad \text{; Power transferred from rotor into the air; vector form.} \tag{14}$$

$$P_{\mathrm{ra}} = -F_{\mathrm{t}}(-V) = F_{\mathrm{t}}V \quad \text{; Scalar form. Positive power. Substitute } F_{\mathrm{t}} \text{ equation:} \tag{15}$$

$$P_{\mathrm{ra}} = \rho S \Delta V V^2 \quad \text{; Power transferred from rotor into the air, } P_{\mathrm{ra}} = f(V, \Delta V). \tag{16}$$

Substituting the given equation for V into the equations for F_{t} and P_{ra} yields:

$$V = \frac{1}{2}\Delta V \quad \text{; Given equation for } V\text{. Substitute into } F_{\mathrm{t}} \text{ and } P_{\mathrm{ra}} \text{ equations:} \tag{17}$$

$$F_{\mathrm{t}} = \rho S \Delta V (\frac{1}{2}\Delta V) \quad \text{; Substitute } V = \frac{1}{2}\Delta V \text{ into } F_{\mathrm{t}} = f(V, \Delta V). \tag{18}$$

$$F_{\mathrm{t}} = \frac{1}{2}\rho S \Delta V^2 \quad \text{; Thrust force of air acting on rotor, } F_{\mathrm{t}} = f(\Delta V). \tag{19}$$

$$P_{\mathrm{ra}} = \rho S \Delta V (\frac{1}{2}\Delta V)^2 \quad \text{; Substitute } V = \frac{1}{2}\Delta V \text{ into } P_{\mathrm{ra}} = f(V, \Delta V). \tag{20}$$

$$P_{\mathrm{ra}} = \frac{1}{4}\rho S \Delta V^3 \quad \text{; Power transferred from the rotor into the air, } P_{\mathrm{ra}} = f(\Delta V). \tag{21}$$

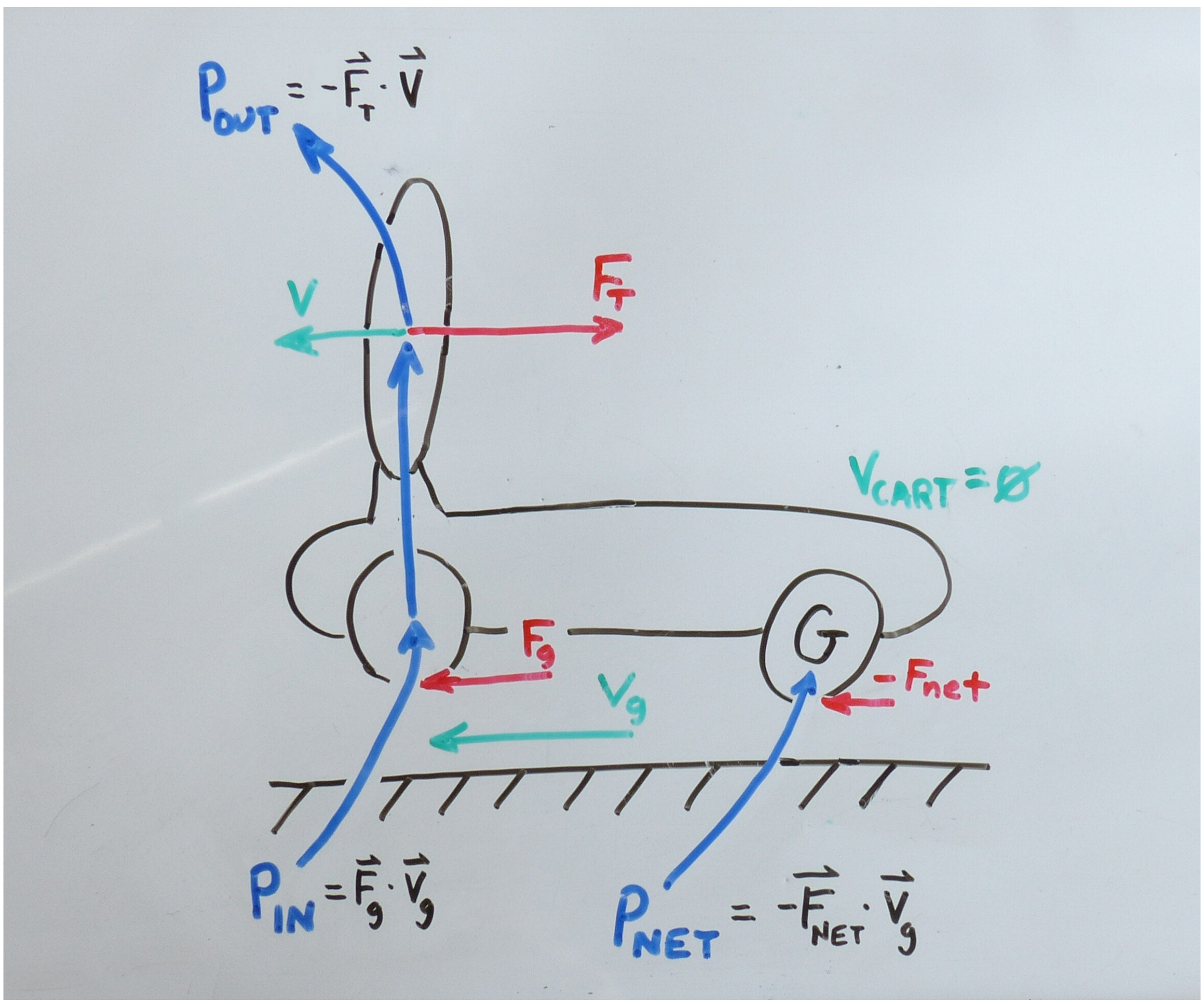

FIG. 3. Power transfers into, through, and out of the cart (shown in blue, cart reference frame).

The propeller is powered by the rear wheels through a 100% efficient transmission, so the power coming in at the wheels, P_{in}, must equal the power going out at the propeller, P_{out}. See figure 3. Applying this principle provides an equation for the force of the ground acting on the rear wheels, F_g:

$$
\begin{aligned}
P_{in} &= P_{out} && \text{; Power in at rear wheels equals power out at the rotor.} && (22)\\
\vec{F}_g \cdot \vec{V}_g &= P_{ra} && \text{; Apply } P = \vec{F}\cdot\vec{V} \text{ to force of ground on rear wheels.} && (23)\\
-F_g(-V_g) &= P_{ra} = F_g V_g && \text{; Scalar form. Solve for } F_g\text{:} && (24)\\
F_g &= \frac{P_{ra}}{V_g} && \text{; Substitute } V_g = V_{wind}\text{:} && (25)\\
F_g &= \frac{P_{ra}}{V_{wind}} && \text{; Substitute } P_{ra} \text{ equation:} && (26)\\
F_g &= \frac{1}{4}\frac{\rho S \Delta V^3}{V_{wind}} && \text{; Force of ground acting on rear wheels } F_g = f(V_{wind}, \Delta V)\text{:} && (27)
\end{aligned}
$$

For static equilibrium, the sum of all the forces acting on the cart in the x direction must equal zero, so:

$$
\begin{aligned}
\sum \vec{F}_x &= 0 && \text{; Condition for static equilibrium.} && (28)\\
0 &= \vec{F}_t + \vec{F}_g + (-\vec{F}_{net}) && \text{; Vector sum of } x \text{ direction forces acting on the cart.} && (29)\\
0 &= F_t - F_g - F_{net} && \text{; Scalar sum of forces. Solve for } F_{net}. && (30)\\
F_{net} &= F_t - F_g && \text{; Net forward force. Substitute } F_t \text{ and } F_g \text{ equations:} && (31)\\
F_{net} &= \frac{1}{2}\rho S \Delta V^2 - \frac{1}{4}\frac{\rho S \Delta V^3}{V_{wind}} && \text{; Factor out common } \tfrac{1}{2}\rho S \text{ term:} && (32)\\
F_{net} &= \frac{1}{2}\rho S(\Delta V^2 - \frac{1}{2}\frac{\Delta V^3}{V_{wind}}) && \text{; Net forward force, } F_{net} = f(V_{wind}, \Delta V). && (33)
\end{aligned}
$$

The cart harvests wind power at the front wheels, where a force equal and opposite to F_{net} is acting in the rearwards direction at a speed equal to $V_g = V_{cart} = V_{wind}$. Applying the general vector equation for power, $P = \vec{F} \cdot \vec{V}$, yields an equation for the net power:

$$
\begin{aligned}
P_{net} &= -\vec{F}_{net} \cdot \vec{V}_g && \text{; } P = \vec{F}\cdot\vec{V} \text{ applied to force at front wheels; vector form.} && (34)\\
P_{net} &= -F_{net}(-V_g) = F_{net} V_g && \text{; Scalar form. Positive power. Substitute } V_g = V_{wind}\text{:} && (35)\\
P_{net} &= F_{net} V_{wind} && \text{; Substitute } F_{net} = f(V_{wind}, \Delta V)\text{:} && (36)\\
P_{net} &= \frac{1}{2}\rho S(\Delta V^2 - \frac{1}{2}\frac{\Delta V^3}{V_{wind}})V_{wind} && \text{; Simplify:} && (37)\\
P_{net} &= \frac{1}{2}\rho S(V_{wind}\Delta V^2 - \frac{1}{2}\Delta V^3) && \text{; Net power harvested from the wind, } P_{net} = f(V_{wind}, \Delta V). && (38)
\end{aligned}
$$

To normalize the equations into dimensionless form, we solve the given axial induction factor equation for ΔV, and substitute that term into the equations. First for F_t:

$$
\begin{aligned}
a &= \frac{1}{2}\frac{\Delta V}{V_{wind}} && \text{; Given equation for } a. \text{ Re-arrange:} && (39)\\
\Delta V &= 2aV_{wind} && \text{; } \Delta V = f(V_{wind}, a). \text{ Substitute into } F_t\text{:} && (40)\\
F_t &= \frac{1}{2}\rho S(2aV_{wind})^2 && \text{; } F_t = f(V_{wind}, a). \text{ Gather } V_{wind} \text{ terms:} && (41)\\
&= \frac{1}{2}\rho S V_{wind}^2 \cdot 4a^2 && \text{; Notice that first term is equal to } F_{wind}\text{, so:} && (42)\\
F_t &= F_{wind} \cdot CF_t && \text{; Generic equation for thrust force, } F_t\text{, where:} && (43)\\
&\left[CF_t = 4a^2\right] && \text{; Answer A1: } CF_t = f(a). && (44)
\end{aligned}
$$

Next substitute $\Delta V = 2aV_{\text{wind}}$ into $P_{\text{ra}} = f(\Delta V)$:

$$P_{\text{ra}} = \frac{1}{4}\rho S(2aV_{\text{wind}})^3 \quad \text{; Substitute } \Delta V = 2aV_{\text{wind}}\text{. Expand:} \tag{45}$$

$$= \frac{1}{4}\rho S 8a^3 V_{\text{wind}}^3 \quad \text{; Cleverly multiply and divide by 2, rearrange:} \tag{46}$$

$$= \frac{1}{2}\rho S V_{\text{wind}}^3 \cdot 4a^3 \quad \text{; Notice that first term is equal to } P_{\text{wind}}\text{, so:} \tag{47}$$

$$P_{\text{ra}} = P_{\text{wind}} \cdot CP_{\text{ra}} \quad \text{; Generic equation for power from rotor into air, } P_{\text{ra}}\text{, where:} \tag{48}$$

$$\left[CP_{\text{ra}} = 4a^3\right] \quad \text{; Answer A2: } CP_{\text{ra}} = f(a)\text{.} \tag{49}$$

Next substitute $\Delta V = 2aV_{\text{wind}}$ into $F_{\text{g}} = f(V_{\text{wind}}, \Delta V)$:

$$F_{\text{g}} = \frac{1}{4}\frac{\rho S(2aV_{\text{wind}})^3}{V_{\text{wind}}} \quad \text{; Substitute } \Delta V = 2aV_{\text{wind}}\text{. Expand:} \tag{50}$$

$$= \frac{1}{4}\rho S 8a^3 V_{\text{wind}}^2 \quad \text{; Multiply and divide by 2, rearrange:} \tag{51}$$

$$= \frac{1}{2}\rho S V_{\text{wind}}^2 \cdot 4a^3 \quad \text{; Notice that first term is equal to } F_{\text{wind}}\text{, so:} \tag{52}$$

$$F_{\text{g}} = F_{\text{wind}} \cdot CF_{\text{g}} \quad \text{; Generic equation for ground force on rear wheels, } F_{\text{g}}\text{, where:} \tag{53}$$

$$\left[CF_{\text{g}} = 4a^3\right] \quad \text{; Answer A3: } CF_{\text{g}} = f(a)\text{. Note that } CF_{\text{g}} = CP_{\text{ra}}\text{.} \tag{54}$$

Plotting the functions for CF_{t} and CF_{g} reveals that the propeller thrust force is greater than the driving ground force at the rear wheels when the axial induction factor value is greater than zero but less than one, i.e. $CF_{\text{t}} > CF_{\text{g}}$ when: $0 < a < 1$. See figure 4.

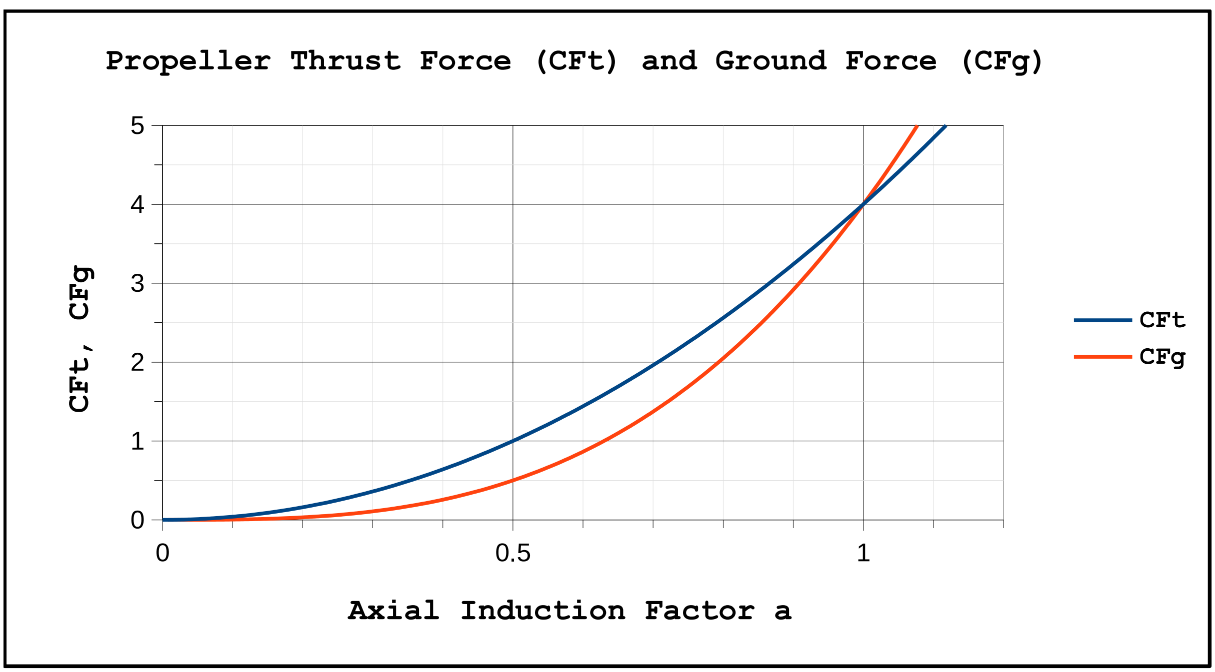

FIG. 4. Plot of functions CF_{t} and CF_{g} vs Axial Induction Factor a.

Next substitute $\Delta V = 2aV_{\text{wind}}$ into $F_{\text{net}} = f(V_{\text{wind}}, \Delta V)$:

$$
\begin{aligned}
F_{\text{net}} &= \frac{1}{2}\rho S\big((2aV_{\text{wind}})^2 - \frac{1}{2}\frac{(2aV_{\text{wind}})^3}{V_{\text{wind}}}\big) && \text{; Substitute } \Delta V = 2aV_{\text{wind}}\text{. Expand:} && (55)\\
&= \frac{1}{2}\rho S(4a^2V_{\text{wind}}^2 - \frac{1}{2}8a^3V_{\text{wind}}^2) && \text{; Factor out } V_{\text{wind}}^2\text{:} && (56)\\
&= \frac{1}{2}\rho SV_{\text{wind}}^2 \cdot (4a^2 - 4a^3) && \text{; Notice that first term is equal to } F_{\text{wind}}\text{, so:} && (57)\\
F_{\text{net}} &= F_{\text{wind}} \cdot CF_{\text{net}} && \text{; Generic equation for net forward force, } F_{\text{net}}\text{, where:} && (58)\\
&\left[CF_{\text{net}} = 4a^2 - 4a^3\right] && \text{; Answer A4: } CF_{\text{net}} = f(a)\text{.} && (59)
\end{aligned}
$$

Next substitute $\Delta V = 2aV_{\text{wind}}$ into $P_{\text{net}} = f(V_{\text{wind}}, \Delta V)$:

$$
\begin{aligned}
P_{\text{net}} &= \frac{1}{2}\rho S\big(V_{\text{wind}}(2aV_{\text{wind}})^2 - \frac{1}{2}(2aV_{\text{wind}})^3\big) && \text{; substitute } \Delta V = 2aV_{\text{wind}}\text{. Expand:} && (60)\\
&= \frac{1}{2}\rho S(4a^2V_{\text{wind}}^3 - 4a^3V_{\text{wind}}^3) && \text{; Factor out } V_{\text{wind}}^3\text{:} && (61)\\
&= \frac{1}{2}\rho SV_{\text{wind}}^3 \cdot (4a^2 - 4a^3) && \text{; Notice that first term is equal to } P_{\text{wind}}\text{, so:} && (62)\\
P_{\text{net}} &= P_{\text{wind}} \cdot CP_{\text{net}} && \text{; Generic equation for net wind power extracted, } P_{\text{net}}\text{, where:} && (63)\\
&\left[CP_{\text{net}} = 4a^2 - 4a^3\right] && \text{; Answer A5: } CP_{\text{net}} = f(a)\text{. Note that } CP_{\text{net}} = CF_{\text{net}}\text{.} && (64)
\end{aligned}
$$

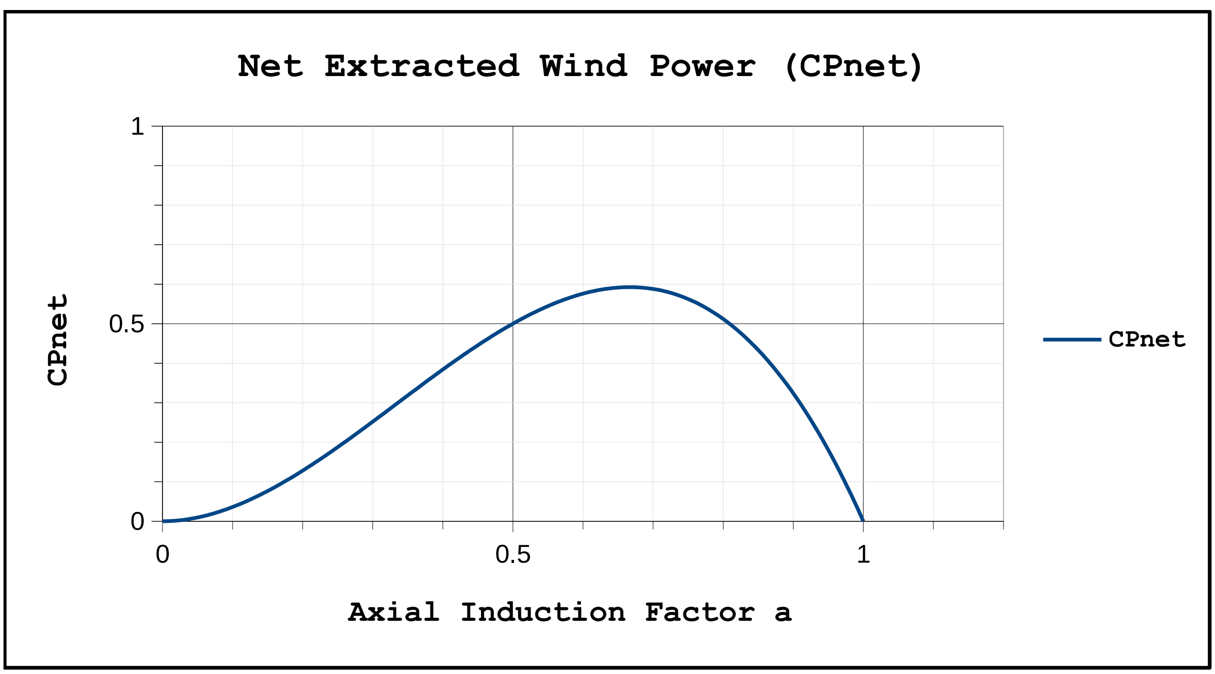

FIG. 5. Plot of function CP_{net} vs Axial Induction Factor a.

Figure 5 shows a plot of CPnet. To determine the maximum possible net power coefficient, we take the derivative of the $CP_{\text{net}} = f(a)$ equation with respect to a, set the resulting equation equal to zero, then solve for the roots:

$$CP_{\text{net}} = 4a^2 - 4a^3 \quad ; CP_{\text{net}} = f(a). \text{ Rearrange:} \tag{65}$$

$$CP_{\text{net}} = -4a^3 + 4a^2 \quad ; \text{Take the derivative with respect to } a\text{:} \tag{66}$$

$$\frac{d(CP_{\text{net}})}{da} = -12a^2 + 8a \quad ; \text{Derivative of } CP_{\text{net}}. \text{ Set equal to 0 and divide by -4:} \tag{67}$$

$$0 = 3a^2 - 2a \quad ; \text{Factor out } a\text{:} \tag{68}$$

$$0 = a(3a - 2) \quad ; \text{Second root is: } 3a - 2 = 0,\ 3a = 2, \text{ and } a = 2/3. \tag{69}$$

$$roots = 0, \frac{2}{3} \quad ; \text{Root values: } 0 = a_{\text{netmin}} \text{ and } 2/3 = a_{\text{netmax}}. \tag{70}$$

$$\left[a_{\text{netmax}} = \frac{2}{3}\right] \quad ; \text{Answer B2: Value of } a \text{ which maximizes } CP_{\text{net}}. \tag{71}$$

Plugging a_{netmax} into $CP_{\text{net}} = f(a)$ reveals the maximum possible net power coefficient:

$$CP_{\text{net}} = 4a^2 - 4a^3 \quad ; CP_{\text{net}} = f(a). \text{ Plug in } a_{\text{netmax}} = 2/3\text{:} \tag{72}$$

$$CP_{\text{netmax}} = 4(\frac{2}{3})^2 - 4(\frac{2}{3})^3 \quad ; \text{Expand:} \tag{73}$$

$$= 4(\frac{4}{9}) - 4(\frac{8}{27}) \quad ; \tag{74}$$

$$= \frac{16}{9} - \frac{32}{27} \quad ; \tag{75}$$

$$= \frac{48 - 32}{27} \quad ; \tag{76}$$

$$\left[CP_{\text{netmax}} = \frac{16}{27}\right] \quad ; \text{Answer B1: } CP_{\text{netmax}} = 16/27 = \text{Spork Limit.} \tag{77}$$

To compute specific parameter values for an idealized downwind propeller cart traveling at or above wind speed, see the author's free and open source online resource: DDW Idealized Propeller Cart Calculator.